Impedance Transmission Conditions for the Electric Potential across a Highly Conductive Casing

Hélène Barucq, Aralar Erdozain, David Pardo, Victor Péron

Impedance Transmission Conditions for the Electric Potential across a Highly Conductive Casing

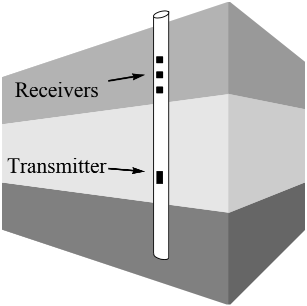
Aralar Erdozain[1], Victor Péron[1,2], Hélène Barucq[1,2] and David Pardo[3,4,5]

[1]INRIA Bordeaux Sud-Ouest, Team magique 3D
[2]Université de Pau et des Pays de l'Adour
[3]University of the Basque Country (UPV/EHU), Leioa, Spain
[4]Basque Center for Applied Mathematics (BCAM), Bilbao, Spain
[5]Ikerbasque (Basque Foundation for Sciences), Bilbao, Spain

INVENTEURS DU MONDE NUMÉRIQUE

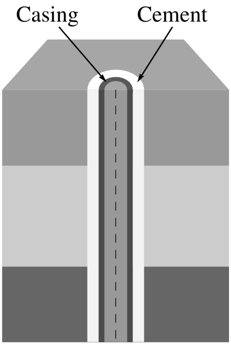
MOTIVATION

- **Main goal:** To obtain a better characterization of the Earth's subsurface
- **How:** Recording borehole resistivity measurements
- **Procedure:**
  - Well
  - Transmitters
  - Receivers



MOTIVATION

- **Practical Difficulties:**
  - It is not easy to drill a borehole
  - It may collapse
- **Practical Solutions:**
  - Use a metallic casing
  - Surround with a cement layer
- **Problem solved, but...**



MOTIVATION

- **Practical Difficulties:**
  - It is not easy to drill a borehole
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- **Practical Solutions:**
  - Use a metallic casing
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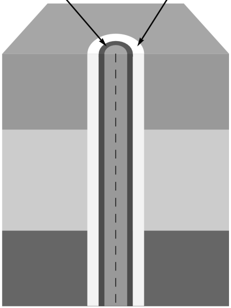
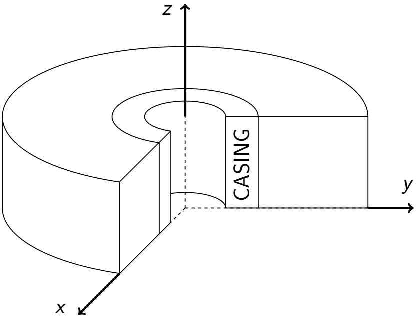


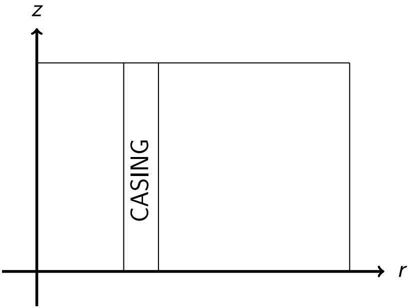
- **Problem solved, but... Numerical problems due to the high conductivity and thinness of the casing**

CONFIGURATION OF INTEREST

SECTIONED 3D DOMAIN



MERIDIAN DOMAIN

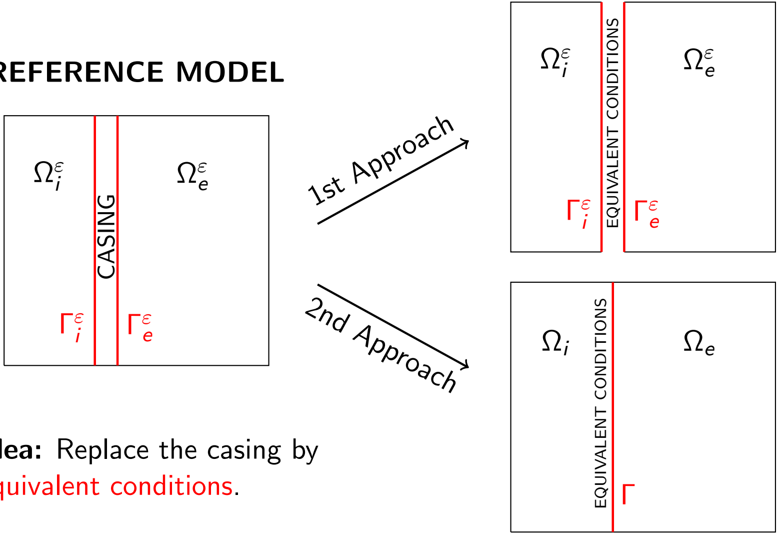


We reduce the 3D problem to a 2D problem due to the axysimmetric configuration.

MAIN IDEA

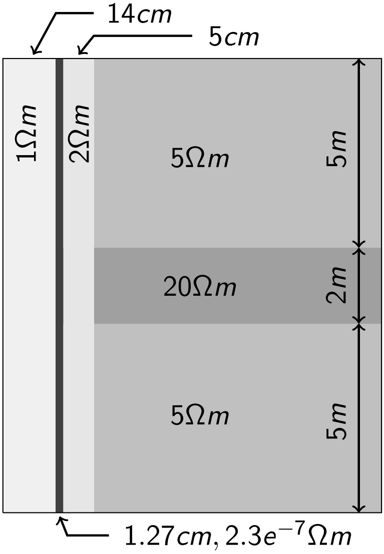
ASYMPTOTIC MODELS

REFERENCE MODEL



Idea: Replace the casing by equivalent conditions.

REALISTIC SCENARIO



- **Conductivity and casing width:**

$$\begin{cases} \varepsilon & = 1.27\text{e} - 2m \\ \sigma_c & = 4.34\text{e}6 \;\; \Omega^{-1} m^{-1} \end{cases}$$

$$\Rightarrow \sigma_c \approx \varepsilon^{-3}$$

- **Classical approach:**

$$\sigma_c = \alpha \qquad \alpha \in \mathbb{R}$$

- **High conductive case:**

$$\sigma_c = \alpha\varepsilon^{-3} \qquad \alpha \in \mathbb{R}$$

REFERENCES

[1] A.A. Kaufman. The electrical field in a borehole with a casing. *Geophysics,* Vol.55, Issue 1, pp. 29-38, 1990.

[2] D.Pardo, C.Torres-Verdín and Z.Zhang. Sensitivity study of borehole-to-surface and crosswell electromagnetic measurements acquired with energized steel casing to water displacement in hydrocarbon-bearing layers. *Geophysics,* 73 No.6, F261-F268, 2008.

[3] M. Duruflé, V. Péron and C. Poignard. Thin Layer Models for Electromagnetism.*Communications in Computational Physics* 16(1):213-238, 2014.

[4] K. Schmidt, A. Chernov, Robust transmission conditions of high order for thin conducting sheets in two dimensions, *IEEE Trans. Magn.*, 50 (2014), pp. 41–44.

OUTLINE

1. Equivalent Conditions
2. Numerical Results
3. Second Approach
4. Application
5. Perspectives

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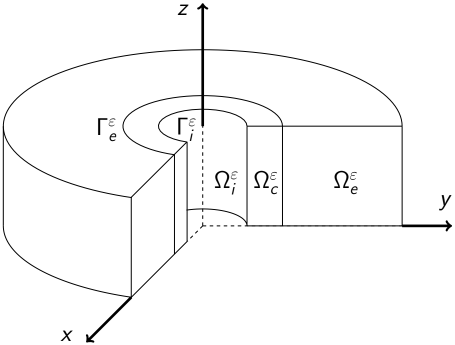
MODEL PROBLEM - 3D

STATIC ELECTRIC POTENTIAL

$$\mathrm{div}\left((\sigma - i\epsilon\omega)\nabla u\right) = f \text{ in } \Omega$$

$$\Omega = \Omega_i^\varepsilon \cup \Omega_c^\varepsilon \cup \Omega_e^\varepsilon \cup \Gamma_i^\varepsilon \cup \Gamma_e^\varepsilon$$



Where f is a known data, σ is piecewise constant

$$\sigma = \begin{cases} \sigma_i & \text{in } \Omega_i^\varepsilon \\ \sigma_c = \alpha\varepsilon^{-3} & \text{in } \Omega_c^\varepsilon \\ \sigma_e & \text{in } \Omega_e^\varepsilon \end{cases}$$

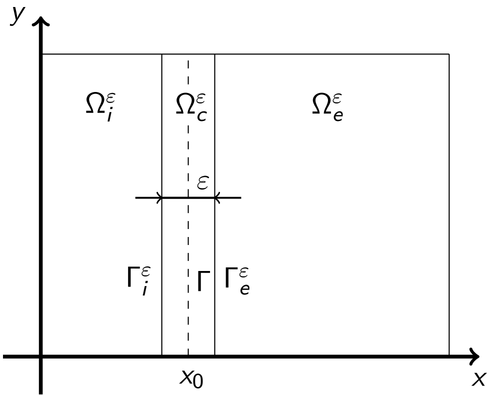
and the solution is expressed as

$$u = \begin{cases} u_i & \text{in } \Omega_i^\varepsilon \\ u_c & \text{in } \Omega_c^\varepsilon \\ u_e & \text{in } \Omega_e^\varepsilon \end{cases}$$

MODEL PROBLEM - 2D

Reference problem (P) set in the domain $\Omega = \Omega_i^\varepsilon \cup \Omega_c^\varepsilon \cup \Omega_e^\varepsilon \cup \Gamma_i^\varepsilon \cup \Gamma_e^\varepsilon$.

We denote the boundary $\Gamma_0 := \left(\overline{\partial\Omega_i^\varepsilon - \Gamma_i^\varepsilon}\right) \cup \left(\overline{\partial\Omega_e^\varepsilon - \Gamma_e^\varepsilon}\right)$.



$$
\begin{cases}
\sigma_i \Delta u_i = f_i & \text{in } \Omega_i^\varepsilon \\
\sigma_c \Delta u_c = 0 & \text{in } \Omega_c^\varepsilon \\
\sigma_e \Delta u_e = f_e & \text{in } \Omega_e^\varepsilon \\
u_i = u_c & \text{on } \Gamma_i^\varepsilon \\
u_c = u_e & \text{on } \Gamma_e^\varepsilon \\
\sigma_i \partial_n u_i = \sigma_c \partial_n u_c & \text{on } \Gamma_i^\varepsilon \\
\sigma_c \partial_n u_c = \sigma_e \partial_n u_e & \text{on } \Gamma_e^\varepsilon \\
u = 0 & \text{on } \partial\Omega
\end{cases}
$$

EXISTENCE AND UNIQUENESS

Theorem 1

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, if $f_i \in L^2(\Omega_i^\varepsilon)$ and $f_e \in L^2(\Omega_e^\varepsilon)$, then $\exists! u \in H_0^1(\Omega)$ solution of (P) and

$$\|u\|_{1,\Omega} \leq C \left(\|f_i\|_{0,\Omega_i^\varepsilon} + \|f_e\|_{0,\Omega_e^\varepsilon} \right).$$

Proof:

- Derive variational formulation, for all $v \in H_0^1(\Omega)$, $a(u, v) = b(v)$.
- Prove that a coercive and continuous, b continuous in $H_0^1(\Omega)$.
- Apply the Lax-Milgram Lema.

DEFINITIONS

Definition 1

We define the jump and mean value of the solution u across the casing as

$$[u] = u|_{\Gamma_e^\varepsilon} - u|_{\Gamma_i^\varepsilon}$$

$$\{u\} = \frac{1}{2}\left(u|_{\Gamma_e^\varepsilon} + u|_{\Gamma_i^\varepsilon}\right)$$

Definition2

Let u be the reference solution. We say an asymptotic model is of Order k+1, if its solution $u^{[k]}$ satisfies

$$||u - u^{[k]}||_{H^1} \leq C\varepsilon^{k+1} \quad \text{when} \quad \varepsilon \to 0$$

METHODOLOGY

- **Step1:** Derive an Asymptotic Expansion for u when $\varepsilon \longrightarrow 0$
  - In the casing: $$u_c(x,y) = \sum_{n\in\mathbb{N}} \varepsilon^n U^n\left(\frac{x-x_0}{\varepsilon}, y\right)$$
  - Outside the casing: $$u(x,y) = \sum_{n\in\mathbb{N}} \varepsilon^n u^n(x,y)$$
- **Step2:** Obtain Equivalent Conditions of order $k+1$ by identifying a simpler problem satisfied by the truncated expansion and neglecting the higher order terms in ε
  - $u_{k,\varepsilon} := u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \ldots + \varepsilon^k u^k$
- **Step3:** Prove Convergence rates for asymptotic models

MULTISCALE EXPANSION

$$
\begin{cases}
\partial_X^2 U^k + \partial_y^2 U^{k-2} = 0 & \text{in} \quad \Omega_c^\varepsilon \\
\sigma_i \Delta u_i^k = f_i & \text{in} \quad \Omega_i^\varepsilon \\
\sigma_e \Delta u_e^k = f_e & \text{in} \quad \Omega_e^\varepsilon \\
u_i^k = U^k & \text{on} \quad \Gamma_i^\varepsilon \\
U^k = u_e^k & \text{on} \quad \Gamma_e^\varepsilon \\
\sigma_i \partial_x u_i^{k-4} = \alpha \partial_X U^k & \text{on} \quad \Gamma_i^\varepsilon \\
\alpha \partial_X U^k = \sigma_e \partial_x u_e^{k-4} & \text{on} \quad \Gamma_e^\varepsilon \\
U^k = 0 & \text{on} \quad \partial\Omega
\end{cases}
$$

PROCEDURE

$$
\begin{tikzcd}
U^0 \arrow[r] & u^0 \arrow[dl] \\
U^1 \arrow[r] & u^1 \arrow[dl] \\
U^2 \arrow[r] & u^2 \arrow[dl] \\
\dots &
\end{tikzcd}
$$

MULTISCALE EXPANSION

$$
\left\{
\begin{array}{rlll}
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U^k & = 0 & \text{on} & \partial\Omega
\end{array}
\right.
$$

PROCEDURE

$$
\begin{tikzcd}
U^0 \arrow[r] & u^0 \arrow[dl] \\
U^1 \arrow[r] & u^1 \arrow[dl] \\
U^2 \arrow[r] & u^2 \arrow[dl] \\
\cdots &
\end{tikzcd}
$$

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\end{cases}
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U^1 \arrow[r] & u^1 \arrow[dl] \\
U^2 \arrow[r] & u^2 \arrow[dl] \\
\dots &
\end{tikzcd}
$$

EQUIVALENT MODELS

FIRST APPROACH

(Transmission conditions across the casing)

- **Order 2:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in} \quad \Omega_i^\varepsilon \\ u_i = 0 & \text{on} \quad \partial\Omega_i^\varepsilon \end{cases}$$
$$\begin{cases} \sigma_e \Delta u_e = f_e & \text{in} \quad \Omega_e^\varepsilon \\ u_e = 0 & \text{on} \quad \partial\Omega_e^\varepsilon \end{cases} \qquad (P_2)$$

- **Order 4:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in} \quad \Omega_i^\varepsilon \\ \sigma_e \Delta u_e = f_e & \text{in} \quad \Omega_e^\varepsilon \\ [u] = 0 & \\ [\sigma \partial_n u] = -\dfrac{\alpha}{\varepsilon^2} \Delta_{\Gamma^\varepsilon} \{u\} & \\ u = 0 & \text{on} \quad \Gamma_0 \end{cases} \qquad (P_4)$$

CONVERGENCE

Theorem 2

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, if $f_i \in L^2(\Omega_i^\varepsilon)$ and $f_e \in L^2(\Omega_e^\varepsilon)$, then $\exists! u^{[k]} \in V_{k+1}$ solution of P_{k+1}, $k = 1, 3$, and

$$\|u^{[k]}\|_{1,\Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C \left(\|f_i\|_{0,\Omega_i^\varepsilon} + \|f_e\|_{0,\Omega_e^\varepsilon} \right)$$

$$\|u - u^{[k]}\|_{1,\Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C\varepsilon^{k+1}$$

Where

$$V_2 = \left\{ v : v_i \in H_0^1\left(\Omega_i^\varepsilon\right), v_e \in H_0^1\left(\Omega_e^\varepsilon\right) \right\}$$

$$V_4 = \left\{ v : v_i \in H^1\left(\Omega_i^\varepsilon\right), v_e \in H^1\left(\Omega_e^\varepsilon\right), \nabla_{\Gamma^\varepsilon}\left\{v\right\} \in L^2\left(\Gamma^\varepsilon\right), v|_{\Gamma_i^\varepsilon} = v|_{\Gamma_e^\varepsilon}, v|_{\Gamma_0} = 0 \right\}$$

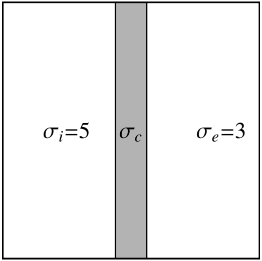
OUTLINE

1. Equivalent Conditions
2. Numerical Results
3. Second Approach
4. Application
5. Perspectives

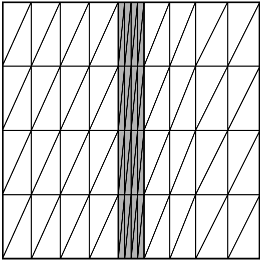
FEM CODE

- Classic Finite Element Method code
- Straight triangular elements (h refinement)
- Lagrange shape functions of any degree (p refinement)

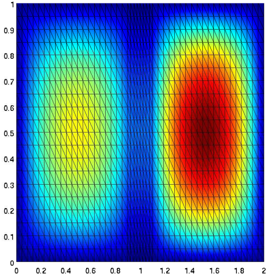
Domain



Mesh

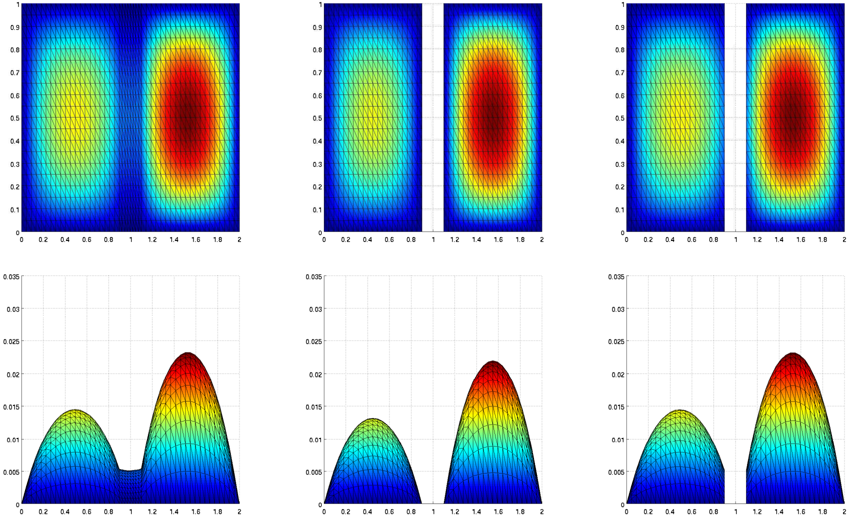


Solution

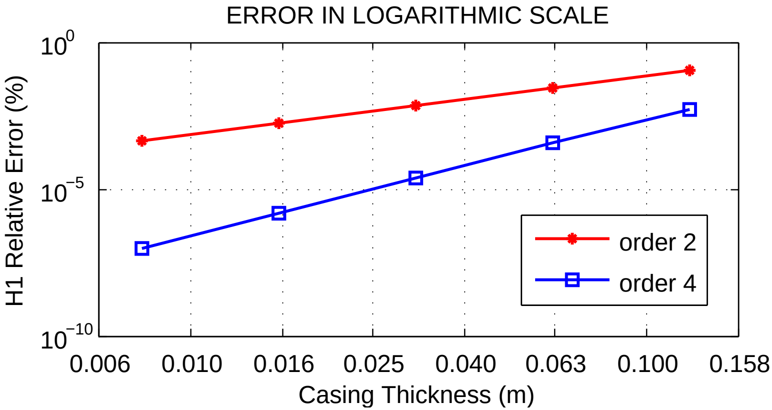


QUALITATIVE COMPARISON

Reference Model **Order 2 Model** **Order 4 Model**



CONVERGENCE RATES



Casing Thickness (ε)	0.0117	0.0234	0.0469	0.0938	Expected ($\varepsilon \to 0$)
Order 2 Slopes	1.9969	1.9899	1.9638	1.8624	2
Order 4 Slopes	4.0046	4.0039	3.9879	3.8992	4

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EQUIVALENT MODELS

SECOND APPROACH

(Transmission conditions across the interface Γ)

- **Order 1:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in} \quad \Omega_i \\ u_i = 0 & \text{on} \quad \partial\Omega_i \end{cases} \qquad\qquad (P_1)$$

$$\begin{cases} \sigma_e \Delta u_e = f_e & \text{in} \quad \Omega_e \\ u_e = 0 & \text{on} \quad \partial\Omega_e \end{cases}$$

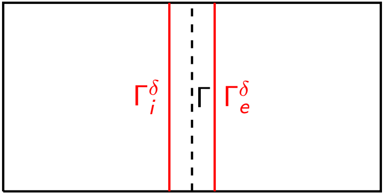
- **Order 2:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in} \quad \Omega_i \\ \sigma_e \Delta u_e = f_e & \text{in} \quad \Omega_e \\ [u] = -\varepsilon \left\{ \partial_n u \right\} & \\ [\partial_n u] = -\dfrac{4}{\varepsilon} \left\{ u \right\} & \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

Stability

Problem

Stability problems for the order 2 model of the second approach due to a non coercive term.



Solution

Use of Artificial Boundaries to move the transmission conditions and recover stability. For $\delta > 0.5$

$$\Gamma_i^\delta = \{(x,y) : x = x_0 - \delta\varepsilon, y \in (0, y_0)\}$$

$$\Gamma_e^\delta = \{(x,y) : x = x_0 + \delta\varepsilon, y \in (0, y_0)\}$$

EQUIVALENT MODELS

SECOND APPROACH

(Transmission conditions across the artificial interfaces)

- **Order 2:**

$$\begin{cases} \sigma_i \Delta u_i = f_i & \text{in} \quad \Omega_i \\ \sigma_e \Delta u_e = f_e & \text{in} \quad \Omega_e \\ [u] = -\varepsilon\,(1-2\delta)\,\{\partial_n u\} & \\ [\partial_n u] = -\dfrac{4\,(1-2\delta)}{\varepsilon}\,\{u\} & \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \qquad (P_2)$$

Stable if $\delta > 0.5$

CONVERGENCE

Theorem 3

There exists $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$, if $f_i \in L^2(\Omega_i^\varepsilon)$ and $f_e \in L^2(\Omega_e^\varepsilon)$, then $\exists! u^{[k]} \in V_{k+1}$ solution of P_{k+1}, $k = 1, 2$, and

$$\|u^{[k]}\|_{1,\Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C \left(\|f_i\|_{0,\Omega_i^\varepsilon} + \|f_e\|_{0,\Omega_e^\varepsilon} \right)$$

$$\|u - u^{[k]}\|_{1,\Omega_i^\varepsilon \cup \Omega_e^\varepsilon} \leq C\varepsilon^{k+1}$$

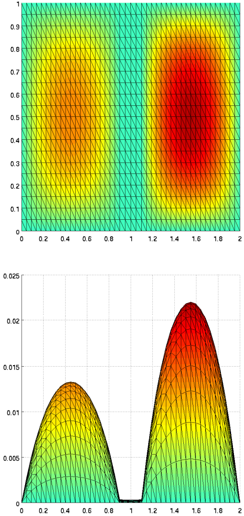
Where

$$V_1 = \left\{ v : v_i \in H_0^1\left(\Omega_i\right), v_e \in H_0^1\left(\Omega_e\right) \right\}$$

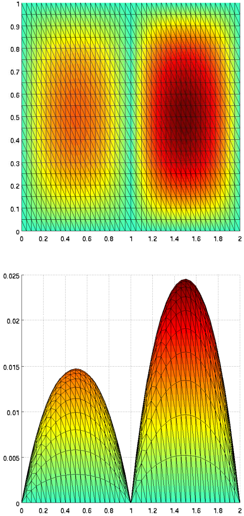
$$V_2 = \left\{ v : v_i \in H^1\left(\Omega_i\right), v_e \in H^1\left(\Omega_e\right), v_i|_\Gamma = v_e|_\Gamma, v|_{\Gamma_0} = 0 \right\}$$

QUALITATIVE COMPARISON

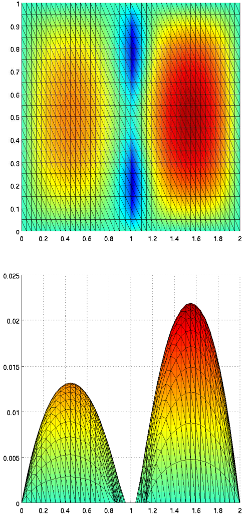
Reference Model



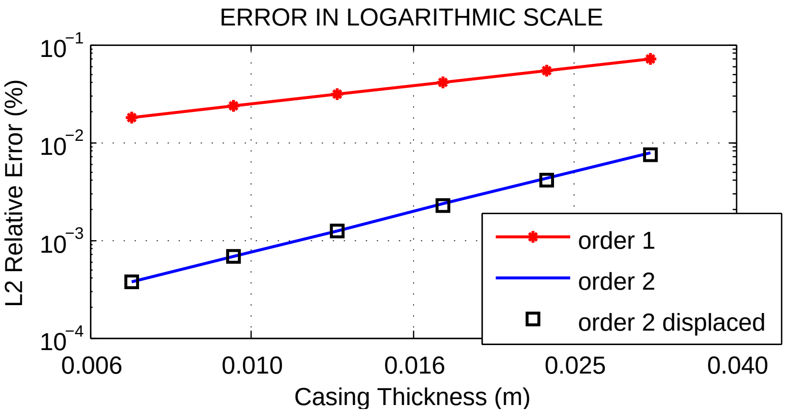
Order 1 Model



Order 2 Model

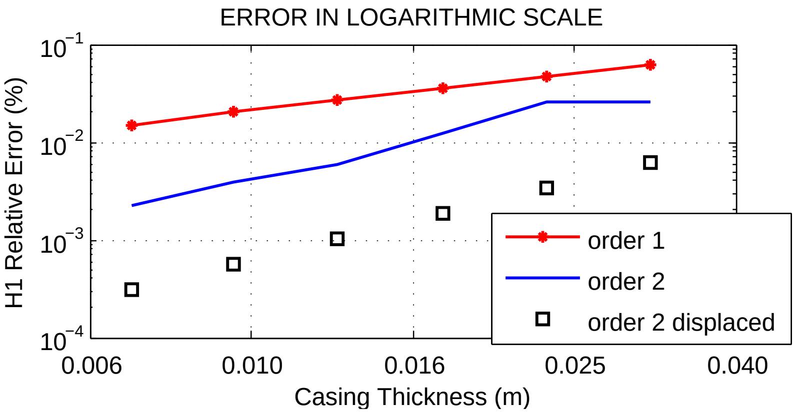


CONVERGENCE RATES L^2



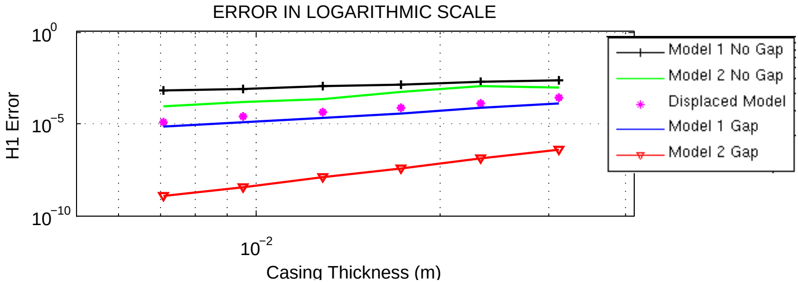
Casing Thickness (ε)	0.008	0.011	0.015	0.020	0.027	Expected ($\varepsilon \to 0$)
Order 1 Slopes	0.972	0.962	0.948	0.928	0.899	1
Order 2 Slopes	2.032	2.011	2.062	2.090	1.9636	2
Order 2 displaced	2.005	2.006	2.007	2.008	2.007	2

CONVERGENCE RATES H^1



Casing Thickness (ε)	0.008	0.011	0.015	0.020	0.027	Expected ($\varepsilon \to 0$)
Order 1 Slopes	0.982	0.975	0.966	0.953	0.935	1
Order 2 Slopes	1.915	1.302	2.463	2.569	-0.125	2
Order 2 displaced	2.004	2.005	2.006	2.008	2.007	2

COMPARISON

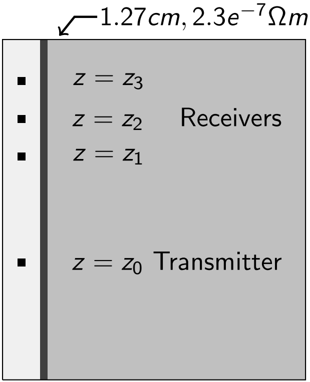


	Order	Stability	ε-independent Domain
Model 1 No Gap	**1**	✓	✓
Model 2 No Gap	**2**	✗	✓
Displaced Model	**2**	✓	✗
Model 1 Gap	**2**	✓	✗
Model 2 Gap	**4**	✓	✗

OUTLINE

1. Equivalent Conditions
2. Numerical Results
3. Second Approach
4. Application
5. Perspectives

APPLICATION



- **Equation:** $\operatorname{div}(\sigma \nabla u) = f$
- **Right hand side:**

$$f = \begin{cases} 1 & \text{In the transmitter} \\ 0 & \text{Outside the transmitter} \end{cases}$$

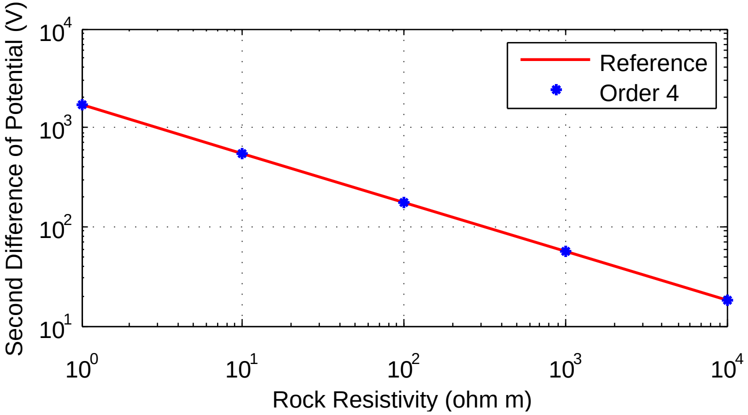
- **Objective:** Measure the second difference of potential on the Receivers

$$U_2 = u(z_1) - 2u(z_2) + u(z_3)$$

- **Expected Result:** (Model of Kaufman) Relation between second difference of potential and rock resistivity of the form

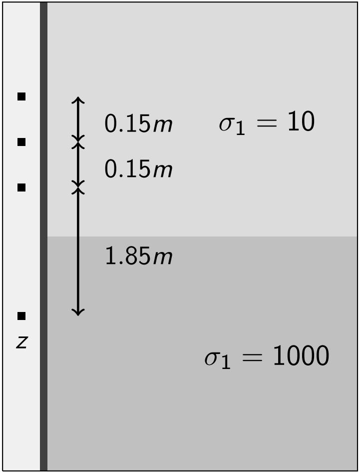
$$U_2 = k \cdot \rho_{\text{rock}}^{-\frac{1}{2}} \qquad k \in \mathbb{R}$$

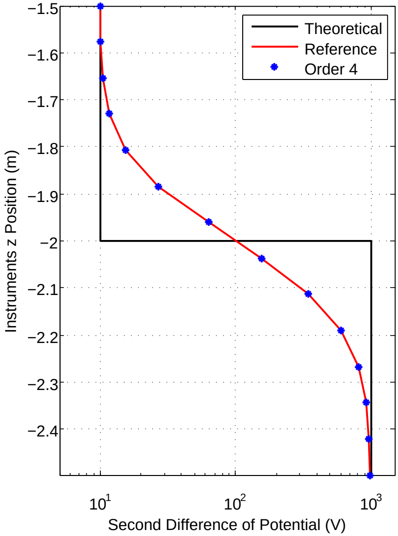
VARYING ROCK CONDUCTIVITY



Resistivity	[1,10]	[10,10^2]	[10^2,10^3]	[10^3,10^4]	Expected
Reference Model Slopes	-0.4914	-0.4924	-0.4978	-0.4991	-0.5
Order 4 Slopes	-0.4914	-0.4924	-0.4977	-0.4993	-0.5

TWO ROCK LAYERS





OUTLINE

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Perspectives

- Obtain semianalytical solutions to reduce the computational cost.
- Consider physically more realistic scenarios.
- Develop 3D electromagnetic models.
- Study highly deviated boreholes.

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THANK YOU FOR YOUR ATTENTION